Lecture 23

Monday, 21 November 2022 11:10 AM

Truthful Mechanisms Without Money.

Problem 1: House Allocation

- n agents A, n houses H
- initially, each agent $a \in A$ is allocated a distinct house $\pi_0(a) \in H$
- agents have strict preferences over houses $>_a$

  $h_1 >_a h_2$: $a \in A$ prefers h_1 to h_2.
- an allocation $\pi : A \to H$ is a bijection

What are DSIC mechanisms for re-allocating houses to agents?
Trivially, each agent could keep her own house.

Top Trading Cycles Algorithm [Gale, Shapley, Scarf '74]

- $i \leftarrow 0$, $A_0 \leftarrow A$,
- $E_i \leftarrow \phi$
- for each agent a in A_i, let h_a be her top choice in the set $\{\pi_0(a)\}_{a \in A_i}$ (possible that $h_a = \pi_0(a)$)
- $E_i \leftarrow \{(a, h_a)\}$ (including self loops)

  note that each $a \in A_i$ has out-degree 1.
- let C_i be a directed cycle in (A_i, E_i)
- for each $a \in C_i$,

  $\pi(a) = h_a$ // we say a is "allocated" by TTCA
- $A_{i+1} \leftarrow A_i \setminus C_i$
- $i \leftarrow i+1$, repeat until $A_{i+1} = \phi$

note that if (i) $\pi(a) = h$, $\exists i : a \in C_i$, $(a, h_a) \in C_i$

(ii) $\forall a \in A_i$, $\pi(a) = \pi_0(a')$ for some $a' \in A_i$

(thus, $\forall a \notin A_i$, $\pi(a) = \pi_0(a')$ for some $a' \notin A_i$)

Defn: Given an allocation π, a <u>blocking coalition</u> $S \subseteq A$ is a subset of agents that can redistribute their initial allocations so that no agent in S is worse off than in π, and at least one agent is strictly better off.

(i.e., $\exists \pi'$: $\pi'(a) >_a \pi(a)$ for some $a \in S$

$\pi'(a) \geq_a \pi(a)$ $\forall a \in S$

and $\pi'(a) \in \{\pi_0(S)\}$)

Defn: An allocation π is <u>in the core</u> if there is no blocking coalition.

Theorem: Let π_0 be the initial allocation, & let π be the allocation produced by TTCA. Then:

(i) π is in the core

(ii) π is the unique core allocation that satisfies $\pi(a) \geq_a \pi_0(a)$

Proof (i) Say π is not in the core, & S is a blocking coalition.

Let π' be an allocation that improves the house allocated to all $a \in S$

Let $a_0 \in S$ be the agent first allocated by TTCA and $\pi'(a_0) \neq \pi(a_0)$. Then it must be that $\pi'(a_0) >_{a_0} \pi(a_0)$, since a_0 is in the blocking coalition.

Say a_0 was allocated in iteration i.

Then $a_0 \in C_i$, $(a_0, \pi(a_0)) \in C_i$, A_i was the remaining set of agents, & $\pi(a_0)$ was a's highest ranked house in $\pi_0(A_i)$ (i.e., houses initially allocated to agents in A_i).

Thus for a_0 to improve her allocation, she must be allocated a house $h = \pi_0(\hat{a})$ for $\hat{a} \notin A_i$

But agent $\hat{a}$ was allocated a house before iteration i.

Hence $\hat{a}$'s allocation cannot be changed in π'.

Hence a_0 cannot be allocated a better house, and a_0 cannot be in the blocking coalition ∎

(ii) Uniqueness: Given π_0, let π' be another core allocation. Will show by induction that $\forall i$, $\forall a \in C_i$, $\pi'(a) = \pi(a)$.

Base case: $i=1$. Suppose $\exists a \in C_1$ s.t. $\pi'(a) \neq \pi(a)$. Note that each agent in C_1 is allocated her top-ranked house in π. Then in π', agents of C_1 form a blocking coalition.

Inductive step: Suppose for $i = 1 \ldots k$, agents in C_i are allocated the same houses in π & π'. Clearly, following the logic of the base case, agents in C_{i+1} must then be allocated the same houses as well. ∎

Theorem: The TTC algorithm is DSIC

Proof: By induction on i.

Problem 2: Kidney Exchange

- n patients need a kidney
- each patient p_i also has a relative d_i who is willing to donate a kidney, but p_i, d_i are incompatible (e.g., diff. blood types)
- thus, n patient-donor couples: (p_1, d_1), (p_2, d_2), ...

pairs of couples could be compatible: p_1, d_2 are compatible, as are p_2, d_1

or even longer chains: (p_1, d_2), (p_2, d_3), (p_3, d_1), ...

can use TTCA here, to obtain such chains... but long chains are complicated. Surgeries have to be done simultaneously, since donors may back out otherwise.

For now, stick to chains of length 2 (pairs of couples).

- For each patient-donor couple i, let W_i be p-d couples that p_i can get a kidney from.

- Problem: find a DSIC mechanism to maximize # of kidney exchanges.

Defn: Mechanism M is DSIC if no couple i that is unmatched when reporting truthfully can get matched to a couple in W_i by misreporting

Consider the graph: $G = (V = [n], E)$

where $\{i, j\} \in E$ if $j \in W_i$ and $i \in W_j$.
- order vertices arbitrarily. Let M_0 be set of <u>all</u> maximum cardinality matchings.
- For $i = 1 \ldots n$
  - Let M_i be set of matchings in M_{i-1} that match couple i.

    if $M_i = \phi$, then $M_i = M_{i-1}$
- Output any matching in M_n

Theorem: (i) M_n is nonempty, max cardinality matching

(ii) Any matching in M_n matches the same set of vertices.

Theorem: The mechanism is DSIC